Week 10. Cauchy Comparison, Lipschitz Regularity, and Finite Propagation

Tran: §1.6 (Theorem 1.23), §1.9 (Theorems 1.36, 1.38), §1.10 (Theorem 1.39)

1. Overview

In this lecture we study three basic properties of viscosity solutions to the Cauchy problem

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{C}$$

2. Learning Objectives

By the end of this lecture, students should be able to:

1. prove the **comparison principle** for the Cauchy problem and hence uniqueness of viscosity solutions;
2. derive the L^∞**-contraction property** and use it to obtain **time Lipschitz continuity**;
3. combine time regularity with coercivity of the Hamiltonian in order to obtain **space Lipschitz continuity**;
4. prove the **finite speed of propagation** property and interpret it as a domain-of-dependence statement.

> **Position in the course.**
> In this course we introduced Hamilton–Jacobi equations through **optimal control and dynamic programming** before studying viscosity theory in full generality. Therefore the results of this week should be viewed not only as abstract PDE theorems, but also as the rigorous well-posedness statements that justify the value-function interpretation.

3. Standing Assumptions

As in Weeks 8–9, D acts on the spatial variable, φ denotes an upper test function, and ψ denotes a lower test function. Contact values are matched by adding constants, which do not change derivatives. Throughout, subsolutions and supersolutions are understood to belong to $\mathrm{BUC}(\mathbb{R}^n \times [0, T])$ on each finite time strip under consideration; boundedness for all $t \geq 0$ at once is not required.

For the comparison principle we assume the structural condition

$$|H(x, p) - H(x, q)| \leq C|p - q|, \qquad |H(x, p) - H(y, p)| \leq C(1 + |p|)\,|x - y| \tag{1.13}$$

for all $x, y, p, q \in \mathbb{R}^n$, for some constant $C > 0$.

Connection with optimal control. In Weeks 2–4, we used

$$H(x, p) = \sup_{v \in V} \{-b(x, v) \cdot p - f(x, v)\},$$

with $|b(x, v)| \leq M_b$ and Lipschitz constants L_b, L_f in x, uniformly in v. Comparing the expressions inside the supremum gives

$$\begin{aligned} |H(x, p) - H(x, q)| &\leq M_b|p - q|, \\ |H(x, p) - H(y, p)| &\leq (L_b|p| + L_f)|x - y|. \end{aligned}$$

Thus (1.13) follows automatically from our control assumptions, with $C = \max\{1, M_b, L_b, L_f\}$. Coercivity, however, is an additional assumption and does not follow from these bounds alone.

For Lipschitz regularity we additionally assume the coercivity condition

$$H \in \mathrm{BUC}(\mathbb{R}^n \times B(0, R)) \text{ for each } R > 0, \qquad \lim_{|p| \to \infty} \inf_{x \in \mathbb{R}^n} H(x, p) = +\infty, \tag{1.33}$$

and the regularity of the initial data

$$u_0 \in C^1(\mathbb{R}^n), \qquad \|u_0\|_{C^1(\mathbb{R}^n)} < \infty. \tag{1.36}$$

4. Extrema at the Terminal Time

Before proving the comparison principle for the Cauchy problem, we need a boundary-in-time lemma.

Lemma 10.1 (Tran Lemma 1.22).
Fix $T > 0$ and let H be continuous. Let $u \in C(\mathbb{R}^n \times [0,T])$ be a viscosity subsolution of

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Assume that $\varphi \in C^1(\mathbb{R}^n \times [0,T])$ and that $u - \varphi$ has a strict maximum at (x_0, T) over $\mathbb{R}^n \times (0,T]$. Then

$$\varphi_t(x_0,T) + H(x_0, D\varphi(x_0,T)) \le 0.$$

Proof

The difficulty is that (x_0,T) is not an interior point in time, so we cannot directly apply the usual viscosity subsolution test. To move the touching point into the interior, define

$$\varphi_\varepsilon(x,t) = \varphi(x,t) + \frac{\varepsilon}{T-t}.$$

Work on a small closed cylinder $\overline{B}_r(x_0) \times [T-r, T]$, with $0 < r < T$. Strictness gives a positive gap on its lateral and bottom boundaries. As $t \uparrow T$, $u - \varphi_\varepsilon \to -\infty$ uniformly on the spatial ball. At the comparison point $(x_0, T - \sqrt{\varepsilon})$, however, the penalty is only $\sqrt{\varepsilon}$. The value there tends to $(u-\varphi)(x_0,T)$, so for small ε the maximum is attained at an interior point $(x_\varepsilon, t_\varepsilon)$. Moreover,

$$(u-\varphi_\varepsilon)(x_0, T-\sqrt{\varepsilon}) \le (u-\varphi)(x_\varepsilon,t_\varepsilon) \le (u-\varphi)(x_0,T).$$

By compactness and strictness of the original maximum, $(x_\varepsilon,t_\varepsilon) \to (x_0,T)$.

Applying the viscosity subsolution inequality at $(x_\varepsilon,t_\varepsilon)$, we obtain

$$(\varphi_\varepsilon)_t(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon, D\varphi_\varepsilon(x_\varepsilon,t_\varepsilon)) \le 0.$$

Now

$$(\varphi_\varepsilon)_t = \varphi_t + \frac{\varepsilon}{(T-t)^2}, \qquad D\varphi_\varepsilon = D\varphi.$$

Hence

$$\varphi_t(x_\varepsilon,t_\varepsilon) + \frac{\varepsilon}{(T-t_\varepsilon)^2} + H(x_\varepsilon, D\varphi(x_\varepsilon,t_\varepsilon)) \le 0.$$

Dropping the nonnegative middle term and passing to the limit $\varepsilon \to 0$, we conclude that

$$\varphi_t(x_0,T) + H(x_0, D\varphi(x_0,T)) \le 0. \qquad \square$$

Remark.
This lemma allows viscosity testing at a terminal-time contact point. The supersolution version follows by using $\psi_\varepsilon = \psi - \varepsilon/(T-t)$. Non-strict contacts are also allowed: add $|x-x_0|^2 + (t-T)^2$ to an upper test, or subtract it from a lower test, without changing the derivatives at the contact point.

5. Comparison Principle for the Cauchy Problem

We now state the main uniqueness theorem.

Theorem 10.2 (Comparison principle for the Cauchy problem; Tran Theorem 1.23). Assume (1.13), and fix $T > 0$. Let $u, v \in \mathrm{BUC}(\mathbb{R}^n \times [0,T])$ be a viscosity subsolution and a viscosity supersolution of

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,T),$$

respectively. Assume also that

$$u(x,0) \le v(x,0) \qquad \text{for all } x \in \mathbb{R}^n.$$

Then

$$u(x,t) \le v(x,t) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T].$$

Proof

Assume by contradiction that

$$\sup_{\mathbb{R}^n \times [0,T]} (u - v) = \sigma > 0.$$

Choose $(x_1, t_1) \in \mathbb{R}^n \times [0,T]$ such that

$$u(x_1,t_1) - v(x_1,t_1) > \frac{3\sigma}{4}.$$

Since $u(\cdot,0) \le v(\cdot,0)$, necessarily $t_1 > 0$.

Choose $\delta > 0$ sufficiently small and then choose $\varepsilon > 0$ sufficiently small so that

$$2\varepsilon |x_1|^2 + 2\delta t_1 < \frac{\sigma}{4}.$$

With δ fixed, define

$$\Phi(x,y,t,s) = u(x,t) - v(y,s) - \frac{|x-y|^2 + (t-s)^2}{\varepsilon^2} - \varepsilon(|x|^2 + |y|^2) - \delta(t+s).$$

Let $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ be a maximum point of Φ. Then

$$\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \ge \Phi(x_1, x_1, t_1, t_1) > \frac{\sigma}{2}.$$

The proof proceeds in six steps.

Step 1. Localization

For each fixed $\varepsilon > 0$, boundedness and the quadratic confinement ensure that the maximum is attained. At the maximum, $\Phi > \sigma/2$ and $\delta(t_\varepsilon + s_\varepsilon) \ge 0$, so

$$\frac{|x_\varepsilon - y_\varepsilon|^2 + (t_\varepsilon - s_\varepsilon)^2}{\varepsilon^2} + \varepsilon(|x_\varepsilon|^2 + |y_\varepsilon|^2) \le \|u\|_\infty + \|v\|_\infty.$$

Bounding each term separately gives, for a constant K independent of ε,

$$|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon| \le K\varepsilon, \qquad |x_\varepsilon| + |y_\varepsilon| \le \frac{K}{\sqrt{\varepsilon}}.$$

Step 2. Vanishing of the penalization gap

Comparing the maximum with $\Phi(x_\varepsilon, x_\varepsilon, t_\varepsilon, t_\varepsilon)$ yields

$$\begin{aligned}\frac{|x_\varepsilon - y_\varepsilon|^2 + (t_\varepsilon - s_\varepsilon)^2}{\varepsilon^2} &\le v(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, s_\varepsilon) \\ &\quad + \varepsilon(|x_\varepsilon|^2 - |y_\varepsilon|^2) + \delta(t_\varepsilon - s_\varepsilon) \longrightarrow 0.\end{aligned}$$

The first term tends to zero by uniform continuity. For the second, use $\varepsilon\big||x_\varepsilon|^2 - |y_\varepsilon|^2\big| \le \varepsilon|x_\varepsilon - y_\varepsilon|(|x_\varepsilon| + |y_\varepsilon|)$ and Step 1. The last term also tends to zero, with δ fixed.

Step 3. Separation from the initial time

We claim that there exists $\mu > 0$ such that, for all sufficiently small ε,

$$t_\varepsilon \geq \mu, \qquad s_\varepsilon \geq \mu.$$

If either time tends to zero along a subsequence, both do by Step 1. Using the initial order at x_ε, we get

$$\begin{aligned} u(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, s_\varepsilon) \leq\ & [u(x_\varepsilon, t_\varepsilon) - u(x_\varepsilon, 0)] \\ & + [v(x_\varepsilon, 0) - v(y_\varepsilon, s_\varepsilon)]. \end{aligned}$$

Both differences tend to zero by uniform continuity, since $|x_\varepsilon - y_\varepsilon| \to 0$. Yet the left-hand side is at least $\max \Phi > \sigma/2$, a contradiction.

Step 4. Subsolution test

Set $M_\varepsilon = \max \Phi$ and fix $(y_\varepsilon, s_\varepsilon)$. The function $u - \varphi$ has maximum zero at $(x_\varepsilon, t_\varepsilon)$, where

$$\varphi(x,t) = v(y_\varepsilon, s_\varepsilon) + \frac{|x - y_\varepsilon|^2 + (t - s_\varepsilon)^2}{\varepsilon^2} + \varepsilon(|x|^2 + |y_\varepsilon|^2) + \delta(t + s_\varepsilon) + M_\varepsilon.$$

Thus $u = \varphi$ at the contact point. If $t_\varepsilon = T$, use Lemma 10.1; otherwise use the interior test. The subsolution inequality gives

$$\frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} + \delta + H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right) \leq 0.$$

Step 5. Supersolution test

Fix $(x_\varepsilon, t_\varepsilon)$ and define the lower test

$$\begin{aligned} \psi(y,s) = u(x_\varepsilon, t_\varepsilon) & - \frac{|x_\varepsilon - y|^2 + (t_\varepsilon - s)^2}{\varepsilon^2} \\ & - \varepsilon(|x_\varepsilon|^2 + |y|^2) - \delta(t_\varepsilon + s) - M_\varepsilon. \end{aligned}$$

Then $v - \psi$ has minimum zero at $(y_\varepsilon, s_\varepsilon)$. Using the supersolution version of Lemma 10.1 if $s_\varepsilon = T$, we obtain

$$\frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} - \delta + H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) \geq 0.$$

Step 6. Subtract and pass to the limit

Set $p_\varepsilon = 2(x_\varepsilon - y_\varepsilon)/\varepsilon^2$. Subtracting the two inequalities cancels the common time derivative and gives

$$2\delta \leq H(y_\varepsilon, p_\varepsilon - 2\varepsilon y_\varepsilon) - H(x_\varepsilon, p_\varepsilon + 2\varepsilon x_\varepsilon).$$

Compare first at the common momentum p_ε. By (1.13), the right-hand side is bounded above by

$$\begin{aligned} & 2C\varepsilon(|x_\varepsilon| + |y_\varepsilon|) + C(1 + |p_\varepsilon|)|x_\varepsilon - y_\varepsilon| \\ & \quad = 2C\varepsilon(|x_\varepsilon| + |y_\varepsilon|) + C|x_\varepsilon - y_\varepsilon| + 2C\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2}. \end{aligned}$$

The first two terms tend to zero by Step 1, and the last by Step 2. No bound on p_ε itself is needed. This contradicts the fixed lower bound $2\delta > 0$. Therefore

$$u \leq v \quad \text{on } \mathbb{R}^n \times [0,T]. \qquad \square$$

Corollary 10.3 (Uniqueness).
Under assumption (1.13), the Cauchy problem (C) has at most one viscosity solution in $\mathrm{BUC}(\mathbb{R}^n \times [0,T])$.

Indeed, for two solutions with the same initial data, Theorem 10.2 gives $u_1 \leq u_2$ and, on reversing their roles, $u_2 \leq u_1$.

6. L^∞-Contraction

The comparison principle immediately implies stability with respect to initial data.

Proposition 10.4 (L^∞-contraction; Tran Exercise 11).
Assume (1.13). Let u_1, u_2 be viscosity solutions of (C) with initial data g_1, g_2, respectively. Then for every $t \geq 0$,

$$\sup_{x\in\mathbb{R}^n} |u_1(x,t) - u_2(x,t)| \leq \sup_{x\in\mathbb{R}^n} |g_1(x) - g_2(x)|.$$

Proof

Let

$$M := \sup_{x\in\mathbb{R}^n} (g_1(x) - g_2(x)).$$

Then

$$g_1(x) \leq g_2(x) + M \qquad \text{for all } x \in \mathbb{R}^n.$$

Since adding a constant does not change the equation, the function $u_2 + M$ is also a viscosity solution. By Theorem 10.2,

$$u_1(x,t) \leq u_2(x,t) + M \qquad \text{for all } (x,t).$$

Hence

$$u_1(x,t) - u_2(x,t) \leq \sup_x (g_1 - g_2).$$

Interchanging the roles of u_1 and u_2, we also obtain

$$u_2(x,t) - u_1(x,t) \leq \sup_x (g_2 - g_1).$$

Therefore

$$\sup_x |u_1(x,t) - u_2(x,t)| \leq \sup_x |g_1(x) - g_2(x)|. \qquad \square$$

Interpretation.
The solution operator for the Cauchy problem is nonexpansive in the L^∞-norm.

7. Lipschitz Regularity

We now prove that, under suitable assumptions, the solution is Lipschitz continuous in both time and space.

Theorem 10.5 (Tran Theorem 1.36).
Assume (1.13), (1.33), and (1.36). Then the Cauchy problem (C) has a unique viscosity solution u, and u is Lipschitz continuous in both space and time.

In particular, there exists $C > 0$ such that

$$|u_t(x,t)| + |Du(x,t)| \leq C \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{1.39}$$

Proof

For existence, we use the Cauchy version of Perron's method (Tran Theorem 1.34 and Remark 1.35), extending the static construction from Week 9. Its barriers are the functions in Step 1 below. Comparison gives uniqueness. We establish the Lipschitz estimate in three steps.

Step 1. Time Lipschitz continuity at $t = 0$

The gradient Du_0 is bounded by (1.36), and (1.33) bounds H uniformly in x on this range of gradients. We may therefore choose

$$C_0 \geq \sup_{x \in \mathbb{R}^n} |H(x, Du_0(x))|.$$

Set $w_\pm(x,t) = u_0(x) \pm C_0 t$. These functions are C^1, and

$$\begin{aligned}(w_-)_t + H(x, Dw_-) &= -C_0 + H(x, Du_0(x)) \leq 0,\\ (w_+)_t + H(x, Dw_+) &= C_0 + H(x, Du_0(x)) \geq 0.\end{aligned}$$

Thus w_- is a subsolution and w_+ is a supersolution, both with initial data u_0. Applying comparison on each finite time strip gives

$$u_0(x) - C_0 t \leq u(x,t) \leq u_0(x) + C_0 t \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Therefore

$$|u(x,t) - u_0(x)| \leq C_0 t.$$

Thus u is Lipschitz in time at the initial time.

Step 2. Time Lipschitz continuity for all $t \geq 0$

Fix $s > 0$, and define

$$v(x,t) := u(x, t+s).$$

Since the Hamiltonian is independent of time, v is also a viscosity solution of the same equation, now with initial data

$$v(x,0) = u(x,s).$$

Applying Proposition 10.4 to u and v, we get

$$\|u(\cdot, t+s) - u(\cdot,t)\|_\infty \leq \|u(\cdot,s) - u(\cdot,0)\|_\infty.$$

Using Step 1 on the right-hand side,

$$\|u(\cdot,s) - u(\cdot,0)\|_\infty \leq C_0 s.$$

Hence

$$\|u(\cdot, t+s) - u(\cdot,t)\|_\infty \leq C_0 s \qquad \text{for all } t, s \geq 0.$$

Therefore u is globally Lipschitz continuous in time.

Step 3. Space Lipschitz continuity

Fix $t_0 > 0$, and let φ touch $u(\cdot, t_0)$ from above at x_0. If necessary, replace φ by $\varphi + |x - x_0|^2$ to make the contact strict; its gradient at x_0 is unchanged. On a small closed cylinder around (x_0, t_0), maximize

$$u(x,t) - \varphi(x) - \frac{(t-t_0)^2}{2\varepsilon}.$$

At a maximizer $(x_\varepsilon, t_\varepsilon)$, compare with (x_ε, t_0). By Step 2,

$$\frac{|t_\varepsilon - t_0|^2}{2\varepsilon} \leq u(x_\varepsilon, t_\varepsilon) - u(x_\varepsilon, t_0) \leq C_0 |t_\varepsilon - t_0|.$$

Thus $t_\varepsilon \to t_0$. Comparing with (x_0, t_0) shows that every limit point of x_ε maximizes $u(\cdot, t_0) - \varphi$ on the closed spatial ball. Strictness gives $x_\varepsilon \to x_0$, so the contact is interior for small ε.

Put $a_\varepsilon = (t_\varepsilon - t_0)/\varepsilon$. Vary only the time by a small increment h. Step 2 and the upper contact give

$$-C_0|h| \le u(x_\varepsilon, t_\varepsilon + h) - u(x_\varepsilon, t_\varepsilon) \le a_\varepsilon h + \frac{h^2}{2\varepsilon}.$$

With ε fixed, divide by $h > 0$ and let $h \downarrow 0$ to obtain $a_\varepsilon \ge -C_0$; taking $h < 0$ similarly gives $a_\varepsilon \le C_0$. The subsolution inequality now yields

$$H(x_\varepsilon, D\varphi(x_\varepsilon)) \le -a_\varepsilon \le C_0.$$

Passing to the limit gives $H(x_0, D\varphi(x_0)) \le C_0$. By uniform coercivity, choose $L > 0$ such that $H(x,p) \le C_0$ implies $|p| \le L$. Thus every spatial upper test has gradient bounded by L.

To obtain the Lipschitz estimate itself, fix y and $K > L$. If

$$u(x,t_0) - u(y,t_0) - K|x-y|$$

has a positive value, its maximum is attained at some $x \ne y$, since the time slice is bounded. The cone $u(y,t_0)+K|x-y|$, shifted by a constant, is then a smooth upper test near that point with gradient magnitude $K > L$, a contradiction. Exchanging x, y and letting $K \downarrow L$ yields

$$|u(x,t_0) - u(y,t_0)| \le L|x-y|.$$

Continuity extends this estimate to $t_0 = 0$. Together with Step 2 this gives

$$|u(x,t) - u(y,s)| \le L|x-y| + C_0|t-s|.$$

By Rademacher's theorem, u is differentiable almost everywhere, so (1.39) follows with $C = L + C_0$. □

8. Global Lipschitz Well-Posedness

The following theorem strengthens Theorem 10.5 by removing the global structural assumption (1.13).

Theorem 10.6 (Tran Theorem 1.38).
Assume (1.33) and (1.36). Then the Cauchy problem (C) has a unique viscosity solution u, and u is Lipschitz continuous in both space and time. In particular, there exists $C > 0$ such that

$$|Du(x,t)| + |u_t(x,t)| \le C \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{1.40}$$

Proof

Choose C_0 as in §7. By uniform coercivity, choose $R > \max\{\|Du_0\|_\infty, C_0+1\}$ so large that $H(x,p) > C_0+1$ whenever $|p| > R$. Let $0 \le \chi \le 1$ be a smooth cutoff equal to 1 on $|p| \le R$ and to 0 on $|p| \ge 2R$, and set

$$\widetilde{H}(x,p) = \chi(p)H(x,p) + (1-\chi(p))|p|.$$

On $|p| > R$ both $H(x,p)$ and $|p|$ exceed C_0, so $\widetilde{H}(x,p) > C_0$. The modified Hamiltonian is uniformly continuous on the whole space by (1.33) and is coercive because it equals $|p|$ for $|p| \ge 2R$.

In Step 6 of §5, the two spatial points approach one another, as do the two test gradients. Global uniform continuity therefore makes the difference of the two $\widetilde{H}$-values tend to zero. This gives comparison for the modified problem, so the same Perron construction applies. Since $\widetilde{H}(x, Du_0) = H(x, Du_0)$, the barriers and the time bound C_0 from §7 are unchanged. Step 3 of §7 gives $\widetilde{H}(x, D\varphi) \le C_0$ for every spatial upper test. Thus $|D\varphi| \le R$, and the cone argument gives an R-Lipschitz solution in space.

At either an upper or a lower contact, fixing the time and taking spatial difference quotients now bounds the test gradient by R. On that ball $\widetilde{H} = H$, so both viscosity inequalities hold for the original equation.

For uniqueness, compare this Lipschitz solution with any other solution in the finite-strip class of §3. The test gradient on the Lipschitz side has norm at most R; the other gradient differs from it by $2\varepsilon(x_\varepsilon + y_\varepsilon)$ up to sign, which tends to zero by §5, Step 1. Both gradients therefore remain in a fixed ball. Uniform continuity of H there gives the same contradiction in either order of comparison, so the two solutions coincide. □

9. Finite Speed of Propagation

We now turn to the final main theorem of this lecture.

Theorem 10.7 (Finite speed of propagation; Tran Theorem 1.39).
Assume (1.13). Let u be a viscosity subsolution and v a viscosity supersolution of

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Assume further that

$$u(x, 0) \le v(x, 0) \qquad \text{for all } x \in B(0, R)$$

for some $R > 0$. Then, with the constant $C > 0$ from (1.13),

$$u(x, t) \le v(x, t) \qquad \text{for all } x \in B(0, R - Ct), \quad 0 \le t \le R/C.$$

Interpretation.
This theorem says that local information in the initial data propagates at finite speed. If two solutions are initially ordered on a ball $B(0, R)$, then the ordering persists for positive times inside the shrinking ball $B(0, R - Ct)$.

In other words, the solution at (x, t) depends only on the initial data in a suitable domain of dependence.

In one space dimension, the theorem produces a literal space-time cone. The interval on which the initial ordering is known shrinks inward at speed C. Points inside the shaded region are still protected from information originating outside $B(0, R)$.

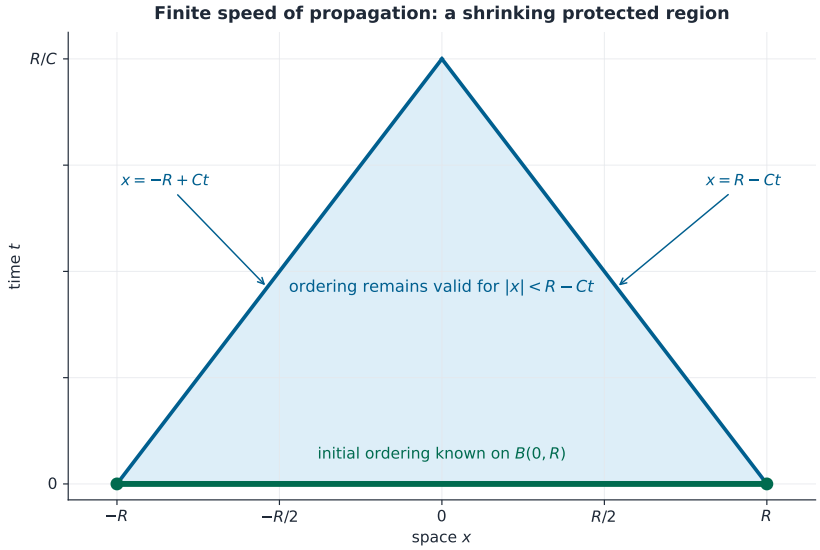


Figure 10.1. If $u(\cdot, 0) \le v(\cdot, 0)$ on $B(0, R)$, then $u(\cdot, t) \le v(\cdot, t)$ on the shrinking ball $B(0, R - Ct)$. The boundary lines have slopes determined by the propagation speed C.

10. Proof Structure of Finite Speed of Propagation

Let

$$w := u - v.$$

The proof begins with a reduction to a simpler first-order inequality.

Lemma 10.8 (Tran Lemma 1.40).
Under assumption (1.13), the function $w = u - v$ is a viscosity subsolution of

$$w_t - C|Dw| = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{1.42}$$

Equivalently, $w_t - C|Dw| \le 0$ in the viscosity sense. Thus the nonlinear comparison problem is reduced to an eikonal-type equation.

Proof

Let φ touch w strictly from above at an interior point (x_0, t_0). On two copies of a small closed cylinder about (x_0, t_0), maximize

$$u(x,t) - v(y,s) - \varphi(x,t) - \frac{|x-y|^2 + (t-s)^2}{\varepsilon^2}.$$

Write $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ for a maximizer. Boundedness on the cylinder first gives $|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon| = O(\varepsilon)$. Comparing with the diagonal point $(x_\varepsilon, x_\varepsilon, t_\varepsilon, t_\varepsilon)$ then gives

$$0 \le \frac{|x_\varepsilon - y_\varepsilon|^2 + (t_\varepsilon - s_\varepsilon)^2}{\varepsilon^2} \le v(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, s_\varepsilon) \longrightarrow 0.$$

Any common limit of the two points must maximize $w - \varphi$ on the cylinder. Strict contact therefore forces both points to converge to (x_0, t_0), so they are interior for small ε.

Set $p_\varepsilon = 2(x_\varepsilon - y_\varepsilon)/\varepsilon^2$ and $a_\varepsilon = 2(t_\varepsilon - s_\varepsilon)/\varepsilon^2$. Fixing $(y_\varepsilon, s_\varepsilon)$ gives an upper test for u; fixing $(x_\varepsilon, t_\varepsilon)$ gives a lower test for v. With the derivatives of φ evaluated at $(x_\varepsilon, t_\varepsilon)$, the two inequalities are

$$\begin{aligned} \varphi_t + a_\varepsilon + H(x_\varepsilon, p_\varepsilon + D\varphi) &\le 0, \\ a_\varepsilon + H(y_\varepsilon, p_\varepsilon) &\ge 0. \end{aligned}$$

Subtract to cancel a_ε. Insert $H(x_\varepsilon, p_\varepsilon)$ between the two Hamiltonians and apply the two bounds in (1.13):

$$\begin{aligned} \varphi_t &\le H(y_\varepsilon, p_\varepsilon) - H(x_\varepsilon, p_\varepsilon + D\varphi) \\ &\le C|D\varphi| + C|x_\varepsilon - y_\varepsilon| + \frac{2C|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2}. \end{aligned}$$

The last two terms vanish, leaving $\varphi_t(x_0,t_0) - C|D\varphi(x_0,t_0)| \le 0$, as required. □

The next ingredient is an explicit barrier.

Lemma 10.9 (Tran Lemma 1.41).
If w is a viscosity subsolution of (1.42) and $w(x,0) \le 0$ on $B(0,R)$, then $w(x,t) \le 0$ whenever $|x| + Ct < R$.

Proof

Set $M = \max\{0, \sup_x w(x,0)\}$. For $0 < \varepsilon < R$, choose a smooth nondecreasing cutoff ξ_ε with $0 \le \xi_\varepsilon \le M$, such that

$$\xi_\varepsilon(s) = 0 \quad (s \le R - \varepsilon), \qquad \xi_\varepsilon(s) = M \quad (s \ge R).$$

Define the comparison barrier

$$\psi_\varepsilon(x,t) = \xi_\varepsilon(|x| + Ct), \qquad 0 \le t < T_\varepsilon := \frac{R-\varepsilon}{C}.$$

For these times the function vanishes near $x = 0$, so it is smooth there despite the appearance of $|x|$. For $x \ne 0$, writing $r = |x| + Ct$ gives $D\psi_\varepsilon = \xi_\varepsilon'(r)x/|x|$. Since the cutoff is nondecreasing,

$$(\psi_\varepsilon)_t - C|D\psi_\varepsilon| = C\xi_\varepsilon' - C|\xi_\varepsilon'| = 0.$$

Also $w(x,0) \le \psi_\varepsilon(x,0)$ everywhere: inside the ball use $w(x,0) \le 0$, and outside use $w(x,0) \le M$. Apply Theorem 10.2 to the reduced Hamiltonian $-C|p|$ on each strip $[0,T]$ with $T < T_\varepsilon$. This gives $w \le \psi_\varepsilon$.

If $|x| + Ct < R$, choose $0 < \varepsilon < R - |x| - Ct$. Then $t < T_\varepsilon$ and $\psi_\varepsilon(x,t) = 0$, so $w(x,t) \le 0$. □

Applying this lemma to $w = u - v$ proves Theorem 10.7.

> **Geometric meaning.**
> The set $B(0, R - Ct)$ is the interior region that still remains protected by the initial ordering at time t. This is the domain-of-dependence picture for the Hamilton–Jacobi flow.

11. Example 10.1: A Burgers-Type One-Dimensional Equation

Consider

$$u_t + \frac{1}{2}(u_x)^2 = 0 \qquad \text{in } \mathbb{R} \times (0,\infty), \qquad u(x,0) = u_0(x).$$

This corresponds to the convex Hamiltonian

$$H(p) = \frac{1}{2}p^2.$$

Observation 1.
The Hamiltonian $H(p) = \frac{1}{2}p^2$ is convex and coercive, but it is **not globally Lipschitz in** p. Indeed, taking $q = 0$ in (1.13) would require $p^2/2 \le C|p|$ for every p.

Observation 2.
Suppose the subsolution and supersolution have a common spatial Lipschitz bound $R > 0$. Every spatial slope of a touching test function then lies in $[-R, R]$. On this interval,

$$|H(p) - H(q)| = \tfrac{1}{2}|p+q|\,|p-q| \le R|p-q|.$$

Extend H by $R|p| - R^2/2$ for $|p| > R$. This extension is globally R-Lipschitz and agrees with H at every test slope. Thus the two viscosity inequalities are unchanged, and Theorem 10.7 gives propagation speed at most R.

Observation 3.
This example prepares for the Hopf–Lax formula, which will be studied later for convex Hamiltonians. For the present week, the point is not to solve the equation explicitly, but to understand that comparison, regularity, and propagation estimates already yield important qualitative information.

> **Preview.**
> If one formally differentiates the equation and sets $w = u_x$, then w satisfies the inviscid Burgers equation
>
> $$w_t + ww_x = 0.$$
>
> This explains heuristically why bounded slopes correspond to bounded propagation speed.

12. Summary

In this lecture we established the basic well-posedness theory for the Cauchy problem for first-order Hamilton–Jacobi equations.

1. The **terminal-time extrema lemma** allows viscosity testing at $t = T$.
2. The **comparison principle** gives uniqueness for the Cauchy problem.
3. The **L^∞-contraction estimate** yields stability with respect to initial data.
4. **Time Lipschitz continuity** follows from a barrier argument and time translation.
5. **Space Lipschitz continuity** follows from coercivity once time regularity is known.
6. The **finite speed of propagation theorem** gives a precise domain-of-dependence statement.

These results form the foundation for the study of convex Hamiltonians and representation formulas in the following weeks.

13. Exercises

Exercise 10.1.
Fill in the details of the proof of Lemma 10.1 by showing that the maximizing point of

$$u - \varphi_\varepsilon$$

converges to (x_0, T) as $\varepsilon \to 0$.

Exercise 10.2.
In the proof of Theorem 10.2, prove carefully that there exists $\mu > 0$ such that

$$t_\varepsilon \geq \mu, \qquad s_\varepsilon \geq \mu$$

for all sufficiently small ε.

Exercise 10.3.
Prove Proposition 10.4 by applying Theorem 10.2 to $u_2 + M$ and u_1.

Exercise 10.4.
Give a detailed proof of Step 3 in Theorem 10.5, namely the spatial Lipschitz estimate obtained from coercivity.

Exercise 10.5.
Explain precisely why the Hamiltonian

$$H(p) = \frac{1}{2}p^2$$

fails to satisfy (1.13), and why uniqueness is still expected in the Lipschitz class.

Exercise 10.6.
Reproduce the proof of finite speed of propagation using the barrier

$$\psi_\varepsilon(x,t) = \xi_\varepsilon(|x| + Ct).$$

14. References

- H. V. Tran, *Hamilton–Jacobi Equations: Theory and Applications*, AMS Graduate Studies in Mathematics 213, 2021.